Principles of Program Analysis:

Data Flow Analysis

Transparencies based on Chapter 2 of the book: Flemming Nielson, Hanne Riis Nielson and Chris Hankin: Principles of Program Analysis. Springer Verlag 2005.

Example Language

Syntax of While-programs

$$
\begin{array}{rcl}
a & ::= & x \mid n \mid a_1 \; op_a \; a_2 \\
b & ::= & \texttt{true} \mid \texttt{false} \mid \texttt{not}\; b \mid b_1 \; op_b \; b_2 \mid a_1 \; op_r \; a_2 \\
S & ::= & [x := a]^\ell \mid [\texttt{skip}]^\ell \mid S_1; S_2 \mid \\
 & & \texttt{if}\; [b]^\ell \;\texttt{then}\; S_1 \;\texttt{else}\; S_2 \mid \texttt{while}\; [b]^\ell \;\texttt{do}\; S
\end{array}
$$

Example: $[\texttt{z:=1}]^1; \texttt{while}\; [\texttt{x>0}]^2 \;\texttt{do}\; ([\texttt{z:=z*y}]^3; [\texttt{x:=x-1}]^4)$

Abstract syntax – parentheses are inserted to disambiguate the syntax

Building an "Abstract Flowchart"

Example: `[z:=1]`^{1}; `while [x>0]`^{2} `do ([z:=z*y]`^{3}; `[x:=x-1]`^{4}`)`

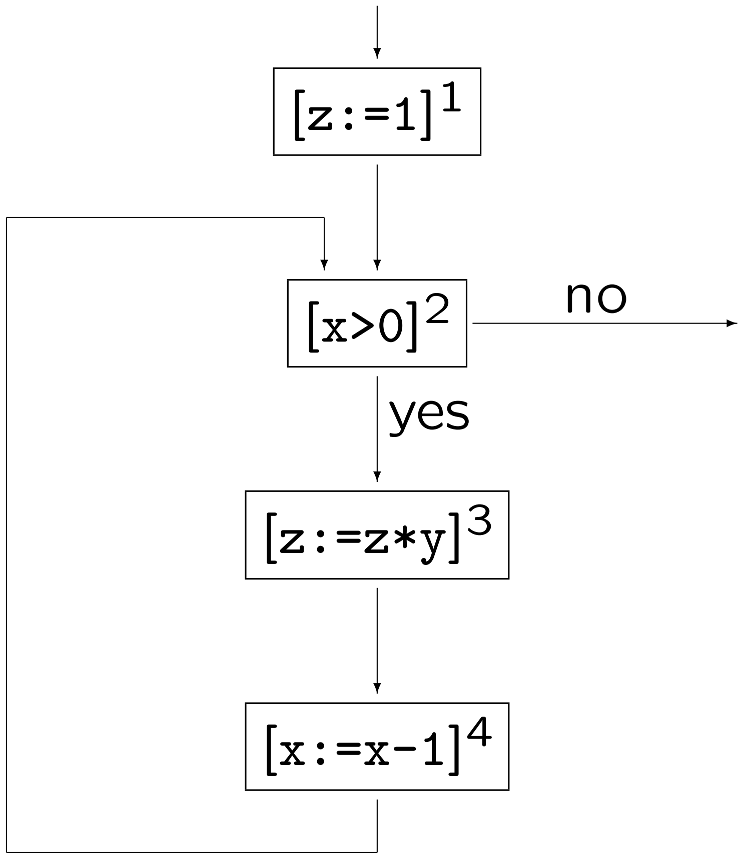
$$init(\cdots) = 1$$

$$final(\cdots) = \{2\}$$

$$labels(\cdots) = \{1, 2, 3, 4\}$$

$$flow(\cdots) = \{(1,2), (2,3), (3,4), (4,2)\}$$

$$flow^R(\cdots) = \{(2,1), (2,4), (3,2), (4,3)\}$$



Initial labels

init(S) is the label of the first elementary block of S:

$$\mathit{init} : \mathbf{Stmt} \rightarrow \mathbf{Lab}$$

$$\begin{array}{rcl}
\mathit{init}([x := a]^\ell) & = & \ell \\
\mathit{init}([\mathtt{skip}]^\ell) & = & \ell \\
\mathit{init}(S_1; S_2) & = & \mathit{init}(S_1) \\
\mathit{init}(\mathtt{if}\ [b]^\ell\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2) & = & \ell \\
\mathit{init}(\mathtt{while}\ [b]^\ell\ \mathtt{do}\ S) & = & \ell
\end{array}$$

Example:

$$\mathit{init}([\mathtt{z{:=}1}]^1; \mathtt{while}\ [\mathtt{x>0}]^2\ \mathtt{do}\ ([\mathtt{z{:=}z*y}]^3; [\mathtt{x{:=}x-1}]^4)) = 1$$

Final labels

final(S) is the set of labels of the last elementary blocks of S:

$$\mathit{final} : \mathbf{Stmt} \to \mathcal{P}(\mathbf{Lab})$$

$$\begin{array}{rcl}
\mathit{final}([x := a]^\ell) & = & \{\ell\} \\
\mathit{final}([\mathtt{skip}]^\ell) & = & \{\ell\} \\
\mathit{final}(S_1; S_2) & = & \mathit{final}(S_2) \\
\mathit{final}(\mathtt{if}\ [b]^\ell\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2) & = & \mathit{final}(S_1) \cup \mathit{final}(S_2) \\
\mathit{final}(\mathtt{while}\ [b]^\ell\ \mathtt{do}\ S) & = & \{\ell\}
\end{array}$$

Example:

$$\mathit{final}([\mathtt{z:=1}]^1; \mathtt{while}\ [\mathtt{x>0}]^2\ \mathtt{do}\ ([\mathtt{z:=z*y}]^3; [\mathtt{x:=x-1}]^4)) = \{2\}$$

Labels

labels(S) is the entire set of labels in the statement S:

$$\textit{labels} : \mathbf{Stmt} \to \mathcal{P}(\mathbf{Lab})$$

$$\begin{array}{rcl}
\textit{labels}([x := a]^\ell) & = & \{\ell\} \\
\textit{labels}([\texttt{skip}]^\ell) & = & \{\ell\} \\
\textit{labels}(S_1; S_2) & = & \textit{labels}(S_1) \cup \textit{labels}(S_2) \\
\textit{labels}(\texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2) & = & \{\ell\} \cup \textit{labels}(S_1) \cup \textit{labels}(S_2) \\
\textit{labels}(\texttt{while } [b]^\ell \texttt{ do } S) & = & \{\ell\} \cup \textit{labels}(S)
\end{array}$$

Example

$$\textit{labels}([\texttt{z:=1}]^1; \texttt{while } [\texttt{x>0}]^2 \texttt{ do } ([\texttt{z:=z*y}]^3; [\texttt{x:=x-1}]^4)) = \{1, 2, 3, 4\}$$

Flows and reverse flows

flow(S) and *flow*$^R(S)$ are representations of how control flows in S:

$$\mathit{flow}, \mathit{flow}^R : \mathbf{Stmt} \to \mathcal{P}(\mathbf{Lab} \times \mathbf{Lab})$$

$$\begin{array}{rcl}
\mathit{flow}([x := a]^\ell) & = & \emptyset \\
\mathit{flow}([\texttt{skip}]^\ell) & = & \emptyset \\
\mathit{flow}(S_1; S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \\
& & \cup \ \{(\ell, \mathit{init}(S_2)) \mid \ell \in \mathit{final}(S_1)\} \\
\mathit{flow}(\texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \\
& & \cup \ \{(\ell, \mathit{init}(S_1)), (\ell, \mathit{init}(S_2))\} \\
\mathit{flow}(\texttt{while } [b]^\ell \texttt{ do } S) & = & \mathit{flow}(S) \cup \{(\ell, \mathit{init}(S))\} \\
& & \cup \ \{(\ell', \ell) \mid \ell' \in \mathit{final}(S)\}
\end{array}$$

$$\mathit{flow}^R(S) = \{(\ell, \ell') \mid (\ell', \ell) \in \mathit{flow}(S)\}$$

Elementary blocks

A statement consists of a set of *elementary blocks*

$$\textit{blocks} : \mathbf{Stmt} \to \mathcal{P}(\mathbf{Blocks})$$

$$\begin{array}{rcl}
\textit{blocks}([\mathtt{x} := a]^\ell) & = & \{[\mathtt{x} := a]^\ell\} \\
\textit{blocks}([\mathtt{skip}]^\ell) & = & \{[\mathtt{skip}]^\ell\} \\
\textit{blocks}(S_1; S_2) & = & \textit{blocks}(S_1) \cup \textit{blocks}(S_2) \\
\textit{blocks}(\mathtt{if}\ [b]^\ell\ \mathtt{then}\ S_1\ \mathtt{else}\ S_2) & = & \{[b]^\ell\} \cup \textit{blocks}(S_1) \cup \textit{blocks}(S_2) \\
\textit{blocks}(\mathtt{while}\ [b]^\ell\ \mathtt{do}\ S) & = & \{[b]^\ell\} \cup \textit{blocks}(S)
\end{array}$$

A statement S is *label consistent* if and only if any two elementary statements $[S_1]^\ell$ and $[S_2]^\ell$ with the same label in S are equal: $S_1 = S_2$

A statement *where all labels are unique* is automatically label consistent

Intraprocedural Analysis

Classical analyses:

- Available Expressions Analysis
- Reaching Definitions Analysis
- Very Busy Expressions Analysis
- Live Variables Analysis

Derived analysis:

- Use-Definition and Definition-Use Analysis

Available Expressions Analysis

The aim of the *Available Expressions Analysis* is to determine

> For each program point, which expressions must have already been computed, and not later modified, on all paths to the program point.

Example:

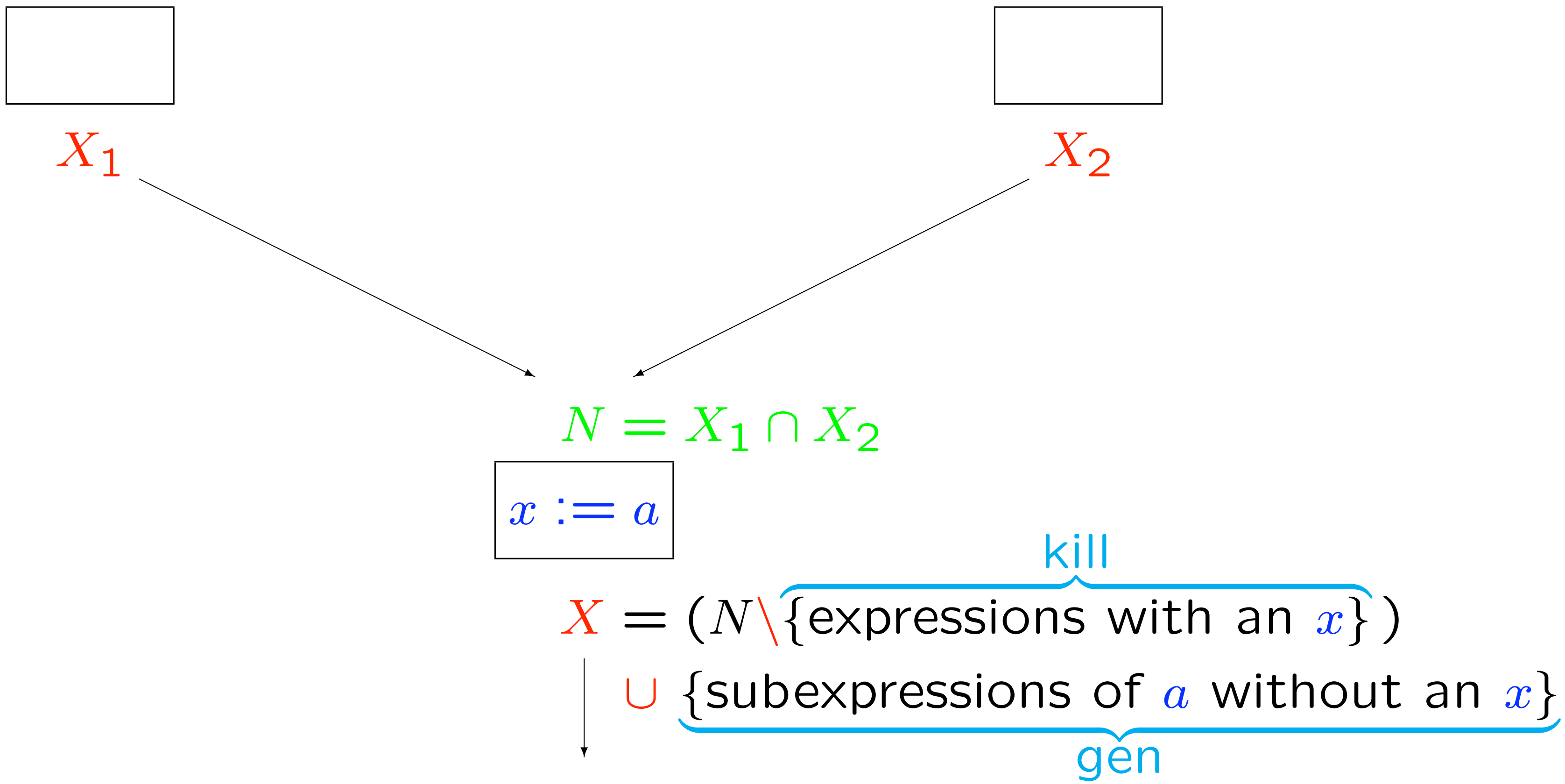
point of interest
⇓

$[\texttt{x:=}\ \texttt{a+b}]^1;\ [\texttt{y:=a*b}]^2;\ \texttt{while}\ [\texttt{y>}\ \texttt{a+b}]^3\ \texttt{do}\ ([\texttt{a:=a+1}]^4;\ [\texttt{x:=}\ \texttt{a+b}]^5)$

The analysis enables a transformation into

$[\texttt{x:=}\ \texttt{a+b}]^1;\ [\texttt{y:=a*b}]^2;\ \texttt{while}\ [\texttt{y>}\ \texttt{x}]^3\ \texttt{do}\ ([\texttt{a:=a+1}]^4;\ [\texttt{x:=}\ \texttt{a+b}]^5)$

Available Expressions Analysis – the basic idea



Available Expressions Analysis

kill and *gen* functions

$$
\begin{aligned}
\mathit{kill}_{\mathsf{AE}}([x := a]^\ell) &= \{a' \in \mathbf{AExp}_\star \mid x \in \mathit{FV}(a')\} \\
\mathit{kill}_{\mathsf{AE}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{kill}_{\mathsf{AE}}([b]^\ell) &= \emptyset \\
\\
\mathit{gen}_{\mathsf{AE}}([x := a]^\ell) &= \{a' \in \mathbf{AExp}(a) \mid x \notin \mathit{FV}(a')\} \\
\mathit{gen}_{\mathsf{AE}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{gen}_{\mathsf{AE}}([b]^\ell) &= \mathbf{AExp}(b)
\end{aligned}
$$

data flow equations: $\mathsf{AE}^=$

$$
\mathsf{AE}_{entry}(\ell) = \begin{cases} \emptyset & \text{if } \ell = \mathit{init}(S_\star) \\ \bigcap\{\mathsf{AE}_{exit}(\ell') \mid (\ell', \ell) \in \mathit{flow}(S_\star)\} & \text{otherwise} \end{cases}
$$

$$
\mathsf{AE}_{exit}(\ell) = (\mathsf{AE}_{entry}(\ell) \backslash \mathit{kill}_{\mathsf{AE}}(B^\ell)) \cup \mathit{gen}_{\mathsf{AE}}(B^\ell)
$$

where $B^\ell \in \mathit{blocks}(S_\star)$

Example:

$$[\texttt{x:=a+b}]^1; [\texttt{y:=a*b}]^2; \texttt{while } [\texttt{y>a+b}]^3 \texttt{ do } ([\texttt{a:=a+1}]^4; [\texttt{x:=a+b}]^5)$$

kill and *gen* functions:

ℓ	$kill_{AE}(\ell)$	$gen_{AE}(\ell)$
1	$\emptyset$	{a+b}
2	$\emptyset$	{a*b}
3	$\emptyset$	{a+b}
4	{a+b, a*b, a+1}	$\emptyset$
5	$\emptyset$	{a+b}

Example (cont.):

$$[\texttt{x:=a+b}]^1;\ [\texttt{y:=a*b}]^2;\ \texttt{while}\ [\texttt{y>a+b}]^3\ \texttt{do}\ ([\texttt{a:=a+1}]^4;\ [\texttt{x:=a+b}]^5)$$

Equations:

$$\begin{aligned}
\mathsf{AE}_{entry}(1) &= \emptyset \\
\mathsf{AE}_{entry}(2) &= \mathsf{AE}_{exit}(1) \\
\mathsf{AE}_{entry}(3) &= \mathsf{AE}_{exit}(2) \cap \mathsf{AE}_{exit}(5) \\
\mathsf{AE}_{entry}(4) &= \mathsf{AE}_{exit}(3) \\
\mathsf{AE}_{entry}(5) &= \mathsf{AE}_{exit}(4)
\end{aligned}$$

$$\begin{aligned}
\mathsf{AE}_{exit}(1) &= \mathsf{AE}_{entry}(1) \cup \{\texttt{a+b}\} \\
\mathsf{AE}_{exit}(2) &= \mathsf{AE}_{entry}(2) \cup \{\texttt{a*b}\} \\
\mathsf{AE}_{exit}(3) &= \mathsf{AE}_{entry}(3) \cup \{\texttt{a+b}\} \\
\mathsf{AE}_{exit}(4) &= \mathsf{AE}_{entry}(4) \backslash \{\texttt{a+b}, \texttt{a*b}, \texttt{a+1}\} \\
\mathsf{AE}_{exit}(5) &= \mathsf{AE}_{entry}(5) \cup \{\texttt{a+b}\}
\end{aligned}$$

Example (cont.):

$[\texttt{x:=a+b}]^1; [\texttt{y:=a*b}]^2; \texttt{while}\ [\texttt{y>}\ \texttt{a+b}\ ]^3\ \texttt{do}\ ([\texttt{a:=a+1}]^4; [\texttt{x:=a+b}]^5)$

Largest solution:

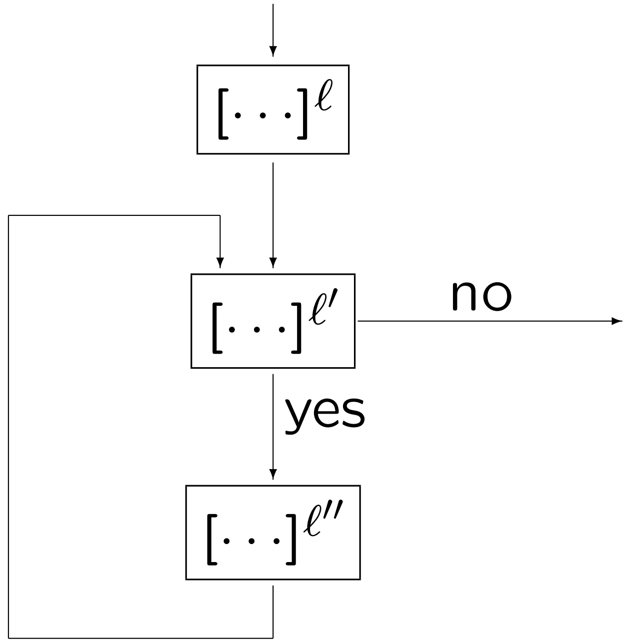
ℓ	$\mathsf{AE}_{entry}(\ell)$	$\mathsf{AE}_{exit}(\ell)$
1	$\emptyset$	$\{$a+b$\}$
2	$\{$a+b$\}$	$\{$a+b, a*b$\}$
3	$\{$a+b$\}$	$\{$a+b$\}$
4	$\{$a+b$\}$	$\emptyset$
5	$\emptyset$	$\{$a+b$\}$

Why largest solution?

$$[\texttt{z:=x+y}]^{\ell};\texttt{while}\ [\texttt{true}]^{\ell'}\ \texttt{do}\ [\texttt{skip}]^{\ell''}$$

Equations:

$$\begin{array}{rcl}
\mathsf{AE}_{entry}(\ell) & = & \emptyset \\
\mathsf{AE}_{entry}(\ell') & = & \mathsf{AE}_{exit}(\ell) \ \cap \ \mathsf{AE}_{exit}(\ell'') \\
\mathsf{AE}_{entry}(\ell'') & = & \mathsf{AE}_{exit}(\ell') \\
\mathsf{AE}_{exit}(\ell) & = & \mathsf{AE}_{entry}(\ell) \cup \{\texttt{x+y}\} \\
\mathsf{AE}_{exit}(\ell') & = & \mathsf{AE}_{entry}(\ell') \\
\mathsf{AE}_{exit}(\ell'') & = & \mathsf{AE}_{entry}(\ell'')
\end{array}$$



After some simplification: $\mathsf{AE}_{entry}(\ell') = \{\texttt{x+y}\} \cap \mathsf{AE}_{entry}(\ell')$

Two solutions to this equation: $\{\texttt{x+y}\}$ and $\emptyset$

Reaching Definitions Analysis

The aim of the *Reaching Definitions Analysis* is to determine

> For each program point, which assignments may have been made and not overwritten, when program execution reaches this point along some path.

Example:

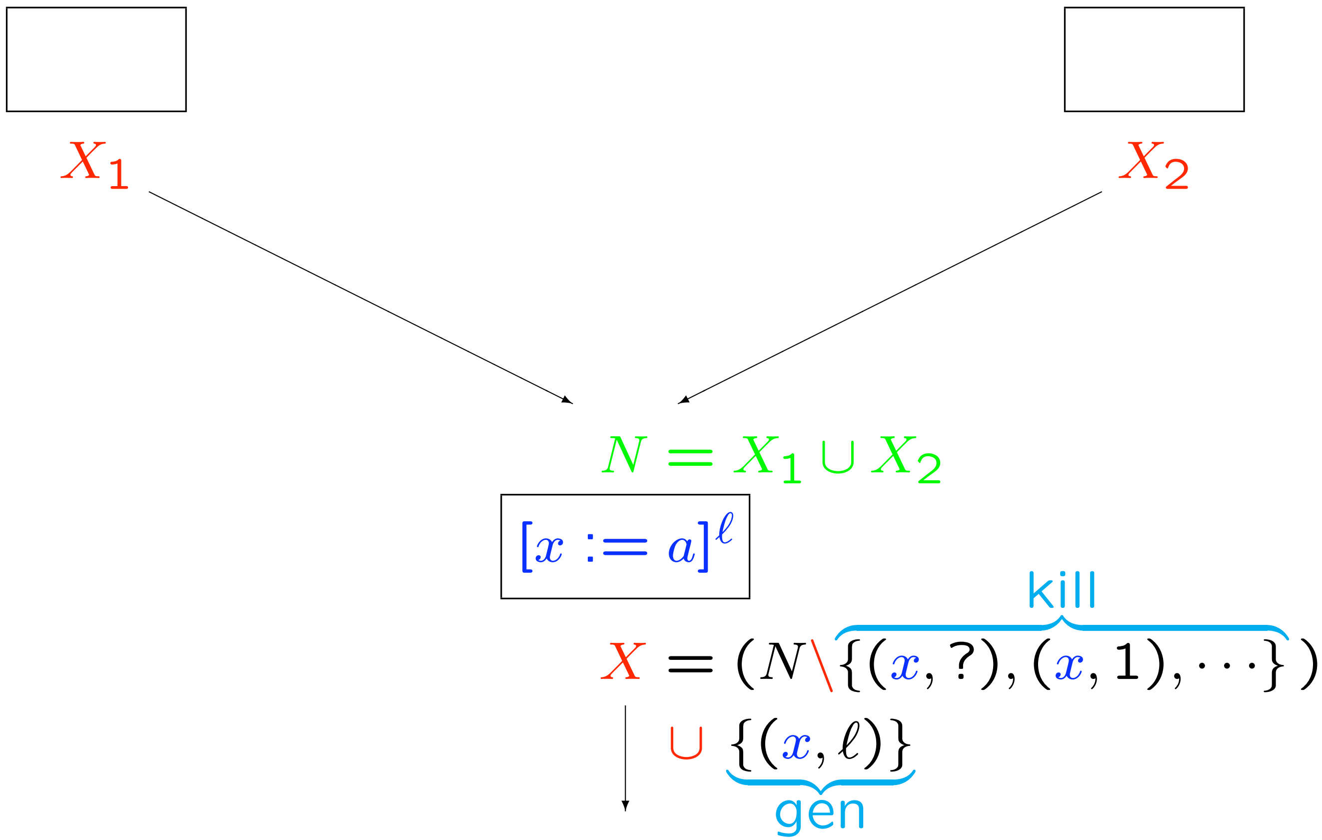
point of interest

$\Downarrow$

$[\mathtt{x:=5}]^1; [\mathtt{y:=1}]^2; \mathtt{while}\ [\mathtt{x>1}]^3\ \mathtt{do}\ ([\mathtt{y:=x*y}]^4; [\mathtt{x:=x-1}]^5)$

useful for definition-use chains and use-definition chains

Reaching Definitions Analysis – the basic idea



Reaching Definitions Analysis

kill and *gen* functions

$$\begin{aligned}
\mathit{kill}_{\mathsf{RD}}([x := a]^\ell) &= \{(x, ?)\} \\
&\quad \cup \{(x, \ell') \mid B^{\ell'} \text{ is an assignment to } x \text{ in } S_\star\} \\
\mathit{kill}_{\mathsf{RD}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{kill}_{\mathsf{RD}}([b]^\ell) &= \emptyset \\
\mathit{gen}_{\mathsf{RD}}([x := a]^\ell) &= \{(x, \ell)\} \\
\mathit{gen}_{\mathsf{RD}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{gen}_{\mathsf{RD}}([b]^\ell) &= \emptyset
\end{aligned}$$

data flow equations: $\mathsf{RD}^=$

$$\mathsf{RD}_{entry}(\ell) = \begin{cases} \{(x, ?) \mid x \in \mathit{FV}(S_\star)\} & \text{if } \ell = \mathit{init}(S_\star) \\ \bigcup\{\mathsf{RD}_{exit}(\ell') \mid (\ell', \ell) \in \mathit{flow}(S_\star)\} & \text{otherwise} \end{cases}$$

$$\mathsf{RD}_{exit}(\ell) = (\mathsf{RD}_{entry}(\ell) \backslash \mathit{kill}_{\mathsf{RD}}(B^\ell)) \cup \mathit{gen}_{\mathsf{RD}}(B^\ell)$$
$$\text{where } B^\ell \in \mathit{blocks}(S_\star)$$

Example:

$$[\texttt{x:=5}]^1; [\texttt{y:=1}]^2; \texttt{while } [\texttt{x>1}]^3 \texttt{ do } ([\texttt{y:=x*y}]^4; [\texttt{x:=x-1}]^5)$$

kill and *gen* functions:

ℓ	$kill_{RD}(\ell)$	$gen_{RD}(\ell)$
1	$\{(x,?),(x,1),(x,5)\}$	$\{(x,1)\}$
2	$\{(y,?),(y,2),(y,4)\}$	$\{(y,2)\}$
3	$\emptyset$	$\emptyset$
4	$\{(y,?),(y,2),(y,4)\}$	$\{(y,4)\}$
5	$\{(x,?),(x,1),(x,5)\}$	$\{(x,5)\}$

Example (cont.):

$$[\texttt{x:=5}]^1;\ [\texttt{y:=1}]^2;\ \texttt{while}\ [\texttt{x>1}]^3\ \texttt{do}\ ([\texttt{y:=x*y}]^4;\ [\texttt{x:=x-1}]^5)$$

Equations:

$$\begin{aligned}
\mathsf{RD}_{entry}(1) &= \{(x,?),(y,?)\}\\
\mathsf{RD}_{entry}(2) &= \mathsf{RD}_{exit}(1)\\
\mathsf{RD}_{entry}(3) &= \mathsf{RD}_{exit}(2) \cup \mathsf{RD}_{exit}(5)\\
\mathsf{RD}_{entry}(4) &= \mathsf{RD}_{exit}(3)\\
\mathsf{RD}_{entry}(5) &= \mathsf{RD}_{exit}(4)
\end{aligned}$$

$$\begin{aligned}
\mathsf{RD}_{exit}(1) &= (\mathsf{RD}_{entry}(1)\backslash\{(x,?),(x,1),(x,5)\}) \cup \{(x,1)\}\\
\mathsf{RD}_{exit}(2) &= (\mathsf{RD}_{entry}(2)\backslash\{(y,?),(y,2),(y,4)\}) \cup \{(y,2)\}\\
\mathsf{RD}_{exit}(3) &= \mathsf{RD}_{entry}(3)\\
\mathsf{RD}_{exit}(4) &= (\mathsf{RD}_{entry}(4)\backslash\{(y,?),(y,2),(y,4)\}) \cup \{(y,4)\}\\
\mathsf{RD}_{exit}(5) &= (\mathsf{RD}_{entry}(5)\backslash\{(x,?),(x,1),(x,5)\}) \cup \{(x,5)\}
\end{aligned}$$

Example (cont.):

$$[\texttt{x:=5}]^1; [\texttt{y:=1}]^2; \texttt{while } [\texttt{x>1}]^3 \texttt{ do } ([\texttt{y:=x*y}]^4; [\texttt{x:=x-1}]^5)$$

Smallest solution:

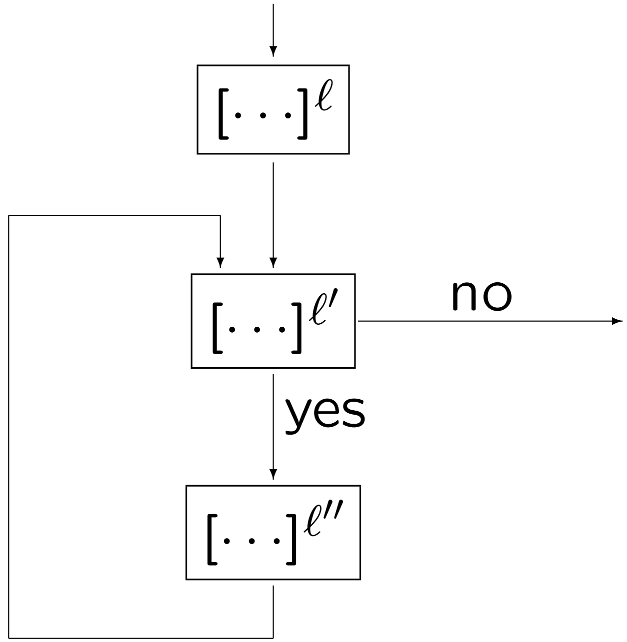
ℓ	$\mathsf{RD}_{entry}(\ell)$	$\mathsf{RD}_{exit}(\ell)$
1	$\{(x,?),(y,?)\}$	$\{(y,?),(x,1)\}$
2	$\{(y,?),(x,1)\}$	$\{(x,1),(y,2)\}$
3	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,2),(y,4),(x,5)\}$
4	$\{(x,1),(y,2),(y,4),(x,5)\}$	$\{(x,1),(y,4),(x,5)\}$
5	$\{(x,1),(y,4),(x,5)\}$	$\{(y,4),(x,5)\}$

Why smallest solution?

$$[\texttt{z:=x+y}]^{\ell}; \texttt{while } [\texttt{true}]^{\ell'} \texttt{ do } [\texttt{skip}]^{\ell''}$$

Equations:

$$\begin{aligned}
\mathsf{RD}_{entry}(\ell) &= \{(\mathtt{x},?),(\mathtt{y},?),(\mathtt{z},?)\}\\
\mathsf{RD}_{entry}(\ell') &= \mathsf{RD}_{exit}(\ell) \cup \mathsf{RD}_{exit}(\ell'')\\
\mathsf{RD}_{entry}(\ell'') &= \mathsf{RD}_{exit}(\ell')\\
\mathsf{RD}_{exit}(\ell) &= (\mathsf{RD}_{entry}(\ell) \setminus \{(\mathtt{z},?)\}) \cup \{(\mathtt{z},\ell)\}\\
\mathsf{RD}_{exit}(\ell') &= \mathsf{RD}_{entry}(\ell')\\
\mathsf{RD}_{exit}(\ell'') &= \mathsf{RD}_{entry}(\ell'')
\end{aligned}$$



After some simplification: $\mathsf{RD}_{entry}(\ell') = \{(\mathtt{x},?),(\mathtt{y},?),(\mathtt{z},\ell)\} \cup \mathsf{RD}_{entry}(\ell')$

Many solutions to this equation: any superset of $\{(\mathtt{x},?),(\mathtt{y},?),(\mathtt{z},\ell)\}$

Very Busy Expressions Analysis

An expression is *very busy* at the exit from a label if, no matter what path is taken from the label, the expression is always used before any of the variables occurring in it are redefined.

The aim of the *Very Busy Expressions Analysis* is to determine

> For each program point, which expressions must be very busy at the exit from the point.

Example:

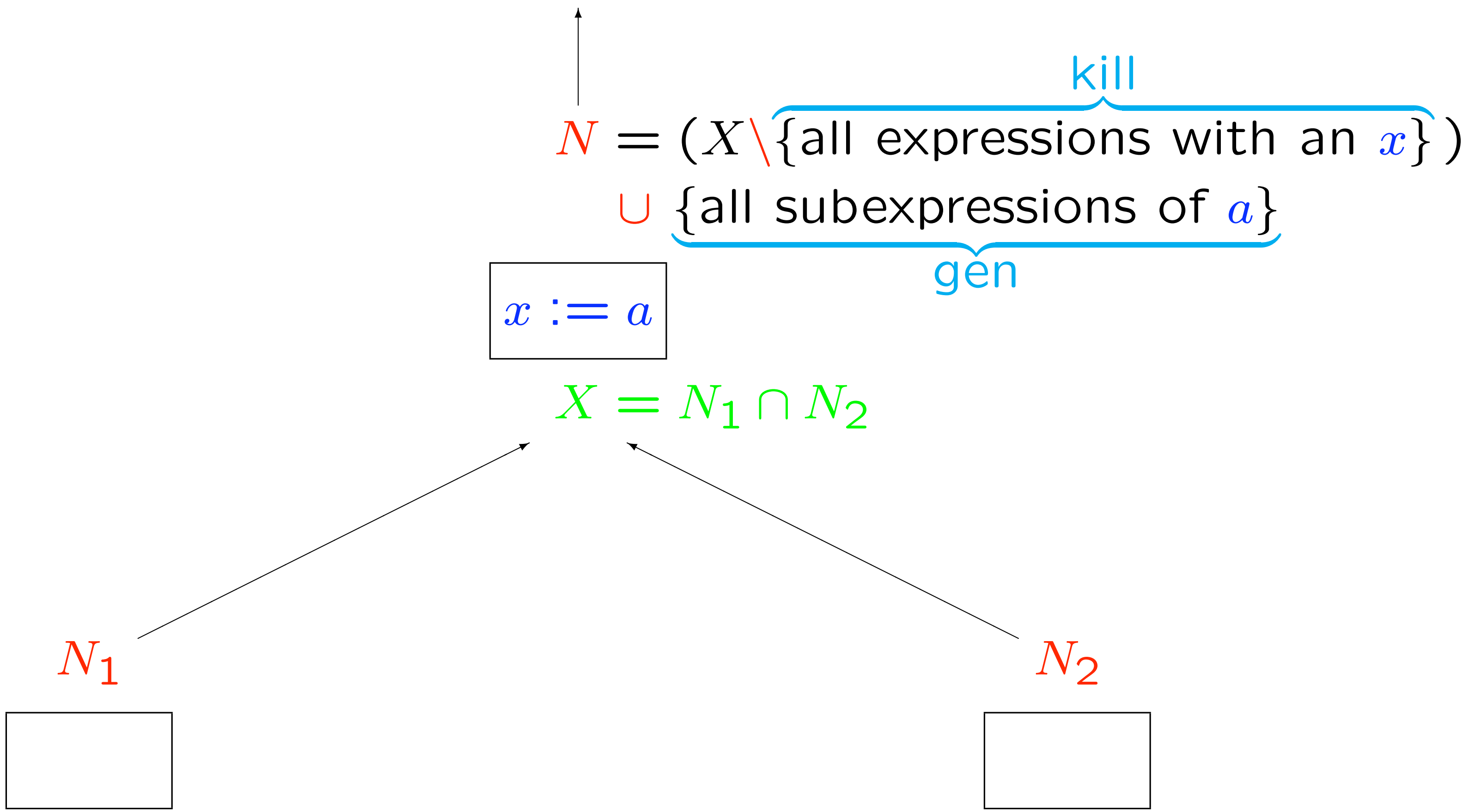
point of interest

$\Downarrow$ `if [a>b]`^{1} `then ([x:=b-a]`^{2}`; [y:=a-b]`^{3}`) else ([y:=b-a]`^{4}`; [x:=a-b]`^{5}`)`

The analysis enables a transformation into

`[t1:=b-a]`^{A}`; [t2:=b-a]`^{B}`;`
`if [a>b]`^{1} `then ([x:=t1]`^{2}`; [y:=t2]`^{3}`) else ([y:=t1]`^{4}`; [x:=t2]`^{5}`)`

Very Busy Expressions Analysis – the basic idea



Very Busy Expressions Analysis

kill and *gen* functions

$$
\begin{aligned}
\mathit{kill}_{\mathsf{VB}}([x := a]^\ell) &= \{a' \in \mathbf{AExp}_\star \mid x \in \mathit{FV}(a')\} \\
\mathit{kill}_{\mathsf{VB}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{kill}_{\mathsf{VB}}([b]^\ell) &= \emptyset \\[6pt]
\mathit{gen}_{\mathsf{VB}}([\mathtt{x} := a]^\ell) &= \mathbf{AExp}(a) \\
\mathit{gen}_{\mathsf{VB}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{gen}_{\mathsf{VB}}([b]^\ell) &= \mathbf{AExp}(b)
\end{aligned}
$$

data flow equations: $\mathsf{VB}^{=}$

$$
\mathsf{VB}_{exit}(\ell) = \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcap\{\mathsf{VB}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}
$$

$$
\mathsf{VB}_{entry}(\ell) = (\mathsf{VB}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{VB}}(B^\ell)) \cup \mathit{gen}_{\mathsf{VB}}(B^\ell)
$$

where $B^\ell \in \mathit{blocks}(S_\star)$

Example:

`if [a>b]`^{1} `then ([x:=b-a]`^{2}`; [y:=a-b]`^{3}`) else ([y:=b-a]`^{4}`; [x:=a-b]`^{5}`)`

kill and *gen* function:

ℓ	$kill_{VB}(\ell)$	$gen_{VB}(\ell)$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	{b-a}
3	$\emptyset$	{a-b}
4	$\emptyset$	{b-a}
5	$\emptyset$	{a-b}

Example (cont.):

if $[a>b]^1$ then $([x:=b-a]^2; [y:=a-b]^3)$ else $([y:=b-a]^4; [x:=a-b]^5)$

Equations:

$$
\begin{aligned}
\mathsf{VB}_{entry}(1) &= \mathsf{VB}_{exit}(1) \\
\mathsf{VB}_{entry}(2) &= \mathsf{VB}_{exit}(2) \cup \{\texttt{b-a}\} \\
\mathsf{VB}_{entry}(3) &= \{\texttt{a-b}\} \\
\mathsf{VB}_{entry}(4) &= \mathsf{VB}_{exit}(4) \cup \{\texttt{b-a}\} \\
\mathsf{VB}_{entry}(5) &= \{\texttt{a-b}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{VB}_{exit}(1) &= \mathsf{VB}_{entry}(2) \cap \mathsf{VB}_{entry}(4) \\
\mathsf{VB}_{exit}(2) &= \mathsf{VB}_{entry}(3) \\
\mathsf{VB}_{exit}(3) &= \emptyset \\
\mathsf{VB}_{exit}(4) &= \mathsf{VB}_{entry}(5) \\
\mathsf{VB}_{exit}(5) &= \emptyset
\end{aligned}
$$

Example (cont.):

$$\texttt{if } [\texttt{a>b}]^1 \texttt{ then } ([\texttt{x:=b-a}]^2; [\texttt{y:=a-b}]^3) \texttt{ else } ([\texttt{y:=b-a}]^4; [\texttt{x:=a-b}]^5)$$

Largest solution:

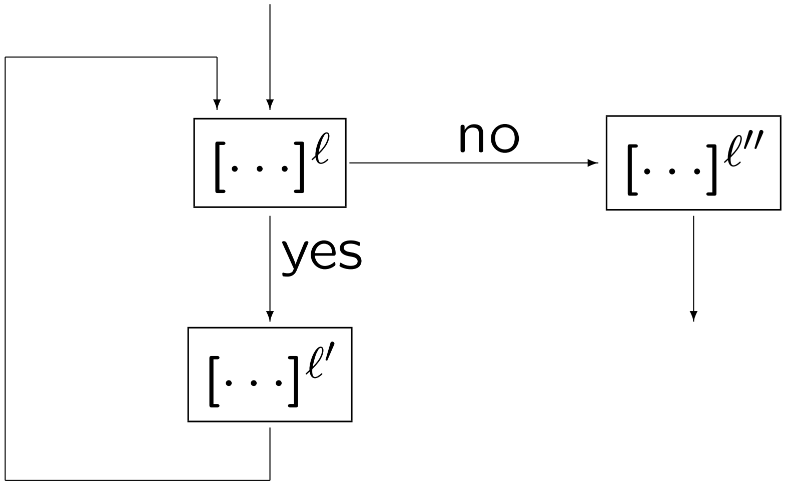
ℓ	$\mathsf{VB}_{entry}(\ell)$	$\mathsf{VB}_{exit}(\ell)$
1	{a-b, b-a}	{a-b, b-a}
2	{a-b, b-a}	{a-b}
3	{a-b}	$\emptyset$
4	{a-b, b-a}	{a-b}
5	{a-b}	$\emptyset$

Why largest solution?

$$(\texttt{while } [\texttt{x>1}]^{\ell} \texttt{ do } [\texttt{skip}]^{\ell'}); [\texttt{x:=x+1}]^{\ell''}$$

Equations:

$$
\begin{aligned}
\mathsf{VB}_{entry}(\ell) &= \mathsf{VB}_{exit}(\ell) \\
\mathsf{VB}_{entry}(\ell') &= \mathsf{VB}_{exit}(\ell') \\
\mathsf{VB}_{entry}(\ell'') &= \{\texttt{x+1}\} \\
\mathsf{VB}_{exit}(\ell) &= \mathsf{VB}_{entry}(\ell') \cap \mathsf{VB}_{entry}(\ell'') \\
\mathsf{VB}_{exit}(\ell') &= \mathsf{VB}_{entry}(\ell) \\
\mathsf{VB}_{exit}(\ell'') &= \emptyset
\end{aligned}
$$



After some simplifications: $\mathsf{VB}_{exit}(\ell) = \mathsf{VB}_{exit}(\ell) \cap \{\texttt{x+1}\}$

Two solutions to this equation: $\{\texttt{x+1}\}$ and $\emptyset$

Live Variables Analysis

A variable is *live* at the exit from a label if there is a path from the label to a use of the variable that does not re-define the variable.

The aim of the *Live Variables Analysis* is to determine

> For each program point, which variables may be live at the exit from the point.

Example:

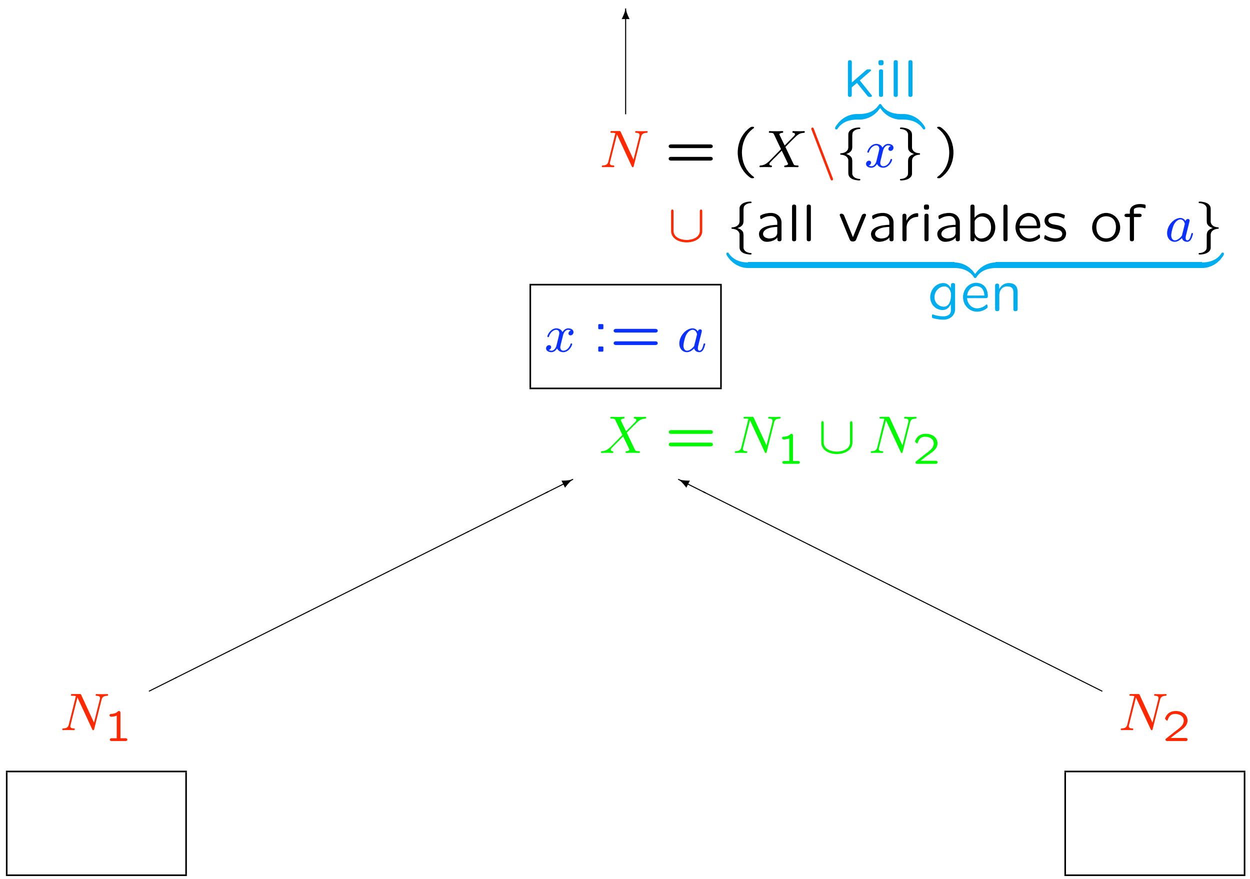
point of interest
⇓

$[\texttt{x}:=2]^1;[\texttt{y}:=4]^2;[\texttt{x}:=1]^3;(\texttt{if }[\texttt{y}>\texttt{x}]^4\texttt{ then }[\texttt{z}:=\texttt{y}]^5\texttt{ else }[\texttt{z}:=\texttt{y*y}]^6);[\texttt{x}:=\texttt{z}]^7$

The analysis enables a transformation into

$$[\texttt{y}:=4]^2;[\texttt{x}:=1]^3;(\texttt{if }[\texttt{y}>\texttt{x}]^4\texttt{ then }[\texttt{z}:=\texttt{y}]^5\texttt{ else }[\texttt{z}:=\texttt{y*y}]^6);[\texttt{x}:=\texttt{z}]^7$$

Live Variables Analysis – the basic idea



Live Variables Analysis

kill and *gen* functions

$$
\begin{aligned}
\mathit{kill}_{\mathsf{LV}}([x := a]^\ell) &= \{x\} \\
\mathit{kill}_{\mathsf{LV}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{kill}_{\mathsf{LV}}([b]^\ell) &= \emptyset
\end{aligned}
$$

$$
\begin{aligned}
\mathit{gen}_{\mathsf{LV}}([x := a]^\ell) &= \mathit{FV}(a) \\
\mathit{gen}_{\mathsf{LV}}([\mathtt{skip}]^\ell) &= \emptyset \\
\mathit{gen}_{\mathsf{LV}}([b]^\ell) &= \mathit{FV}(b)
\end{aligned}
$$

data flow equations: $\mathsf{LV}^{=}$

$$
\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell', \ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}
$$

$$
\mathsf{LV}_{entry}(\ell) = (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}(B^\ell)) \cup \mathit{gen}_{\mathsf{LV}}(B^\ell)
$$

where $B^\ell \in \mathit{blocks}(S_\star)$

Example:

$[x:=2]^1; [y:=4]^2; [x:=1]^3; (\texttt{if } [y>x]^4 \texttt{ then } [z:=y]^5 \texttt{ else } [z:=y*y]^6); [x:=z]^7$

kill and *gen* functions:

ℓ	$kill_{LV}(\ell)$	$gen_{LV}(\ell)$
1	$\{x\}$	$\emptyset$
2	$\{y\}$	$\emptyset$
3	$\{x\}$	$\emptyset$
4	$\emptyset$	$\{x, y\}$
5	$\{z\}$	$\{y\}$
6	$\{z\}$	$\{y\}$
7	$\{x\}$	$\{z\}$

Example (cont.):

$[\mathtt{x:=2}]^1;\ [\mathtt{y:=4}]^2;\ [\mathtt{x:=1}]^3;\ (\mathtt{if}\ [\mathtt{y>x}]^4\ \mathtt{then}\ [\mathtt{z:=y}]^5\ \mathtt{else}\ [\mathtt{z:=y*y}]^6);\ [\mathtt{x:=z}]^7$

Equations:

$$
\begin{aligned}
\mathsf{LV}_{entry}(1) &= \mathsf{LV}_{exit}(1) \backslash \{\mathtt{x}\} \\
\mathsf{LV}_{entry}(2) &= \mathsf{LV}_{exit}(2) \backslash \{\mathtt{y}\} \\
\mathsf{LV}_{entry}(3) &= \mathsf{LV}_{exit}(3) \backslash \{\mathtt{x}\} \\
\mathsf{LV}_{entry}(4) &= \mathsf{LV}_{exit}(4) \cup \{\mathtt{x}, \mathtt{y}\} \\
\mathsf{LV}_{entry}(5) &= (\mathsf{LV}_{exit}(5) \backslash \{\mathtt{z}\}) \cup \{\mathtt{y}\} \\
\mathsf{LV}_{entry}(6) &= (\mathsf{LV}_{exit}(6) \backslash \{\mathtt{z}\}) \cup \{\mathtt{y}\} \\
\mathsf{LV}_{entry}(7) &= \{\mathtt{z}\}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{LV}_{exit}(1) &= \mathsf{LV}_{entry}(2) \\
\mathsf{LV}_{exit}(2) &= \mathsf{LV}_{entry}(3) \\
\mathsf{LV}_{exit}(3) &= \mathsf{LV}_{entry}(4) \\
\mathsf{LV}_{exit}(4) &= \mathsf{LV}_{entry}(5) \cup \mathsf{LV}_{entry}(6) \\
\mathsf{LV}_{exit}(5) &= \mathsf{LV}_{entry}(7) \\
\mathsf{LV}_{exit}(6) &= \mathsf{LV}_{entry}(7) \\
\mathsf{LV}_{exit}(7) &= \emptyset
\end{aligned}
$$

Example (cont.):

$[\texttt{x:=2}]^1; [\texttt{y:=4}]^2; [\texttt{x:=1}]^3; (\texttt{if } [\texttt{y>x}]^4 \texttt{ then } [\texttt{z:=y}]^5 \texttt{ else } [\texttt{z:=y*y}]^6); [\texttt{x:=z}]^7$

Smallest solution:

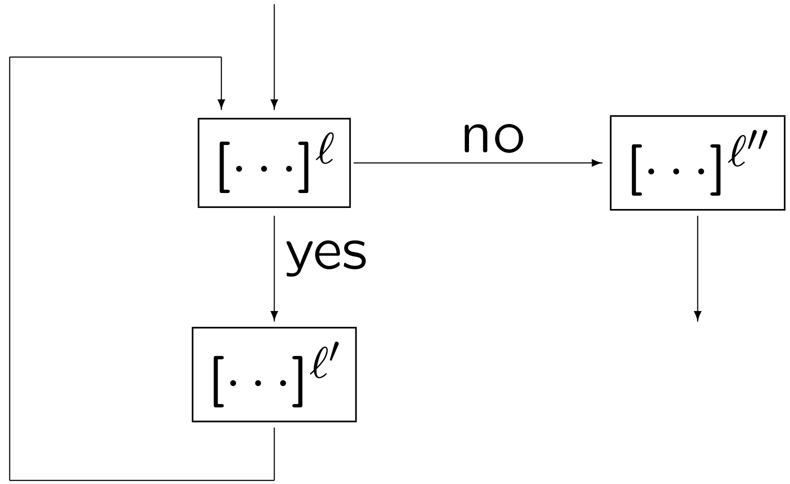
ℓ	$\mathsf{LV}_{entry}(\ell)$	$\mathsf{LV}_{exit}(\ell)$
1	$\emptyset$	$\emptyset$
2	$\emptyset$	$\{\texttt{y}\}$
3	$\{\texttt{y}\}$	$\{\texttt{x},\texttt{y}\}$
4	$\{\texttt{x},\texttt{y}\}$	$\{\texttt{y}\}$
5	$\{\texttt{y}\}$	$\{\texttt{z}\}$
6	$\{\texttt{y}\}$	$\{\texttt{z}\}$
7	$\{\texttt{z}\}$	$\emptyset$

Why smallest solution?

$$(\texttt{while } [\texttt{x>1}]^{\ell} \texttt{ do } [\texttt{skip}]^{\ell'}); [\texttt{x:=x+1}]^{\ell''}$$

Equations:

$$\begin{aligned}
\mathsf{LV}_{entry}(\ell) &= \mathsf{LV}_{exit}(\ell) \cup \{\texttt{x}\} \\
\mathsf{LV}_{entry}(\ell') &= \mathsf{LV}_{exit}(\ell') \\
\mathsf{LV}_{entry}(\ell'') &= \{\texttt{x}\} \\
\mathsf{LV}_{exit}(\ell) &= \mathsf{LV}_{entry}(\ell') \cup \mathsf{LV}_{entry}(\ell'') \\
\mathsf{LV}_{exit}(\ell') &= \mathsf{LV}_{entry}(\ell) \\
\mathsf{LV}_{exit}(\ell'') &= \emptyset
\end{aligned}$$



After some calculations: $\mathsf{LV}_{exit}(\ell) = \mathsf{LV}_{exit}(\ell) \cup \{\texttt{x}\}$

Many solutions to this equation: any superset of $\{\texttt{x}\}$

Derived Data Flow Information

- *Use-Definition chains* or *ud chains*:

  each use of a variable is linked to all assignments that reach it

  $[\mathtt{x}:=0]^1; [\mathtt{x}:=3]^2; (\mathtt{if}\ [\mathtt{z}=\mathtt{x}]^3\ \mathtt{then}\ [\mathtt{z}:=0]^4\ \mathtt{else}\ [\mathtt{z}:=\mathtt{x}]^5); [\mathtt{y}:=\mathtt{x}]^6; [\mathtt{x}:=\mathtt{y}+\mathtt{z}]^7$

- *Definition-Use chains* or *du chains*:

  each assignment to a variable is linked to all uses of it

  $[\mathtt{x}:=0]^1; [\mathtt{x}:=3]^2; (\mathtt{if}\ [\mathtt{z}=\mathtt{x}]^3\ \mathtt{then}\ [\mathtt{z}:=0]^4\ \mathtt{else}\ [\mathtt{z}:=\mathtt{x}]^5); [\mathtt{y}:=\mathtt{x}]^6; [\mathtt{x}:=\mathtt{y}+\mathtt{z}]^7$

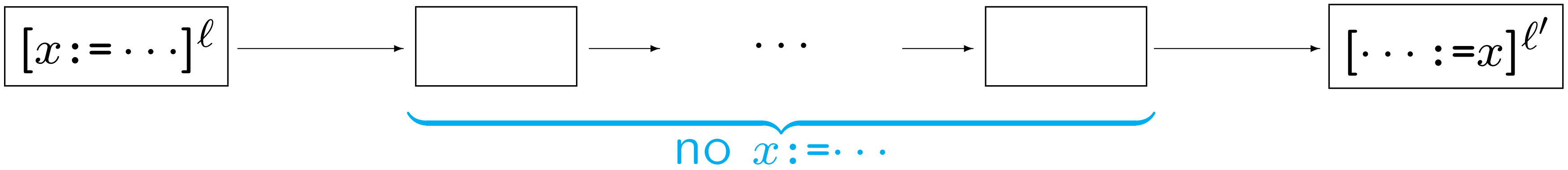
ud chains

$$ud : \mathbf{Var}_\star \times \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Lab}_\star)$$

given by

$$\begin{array}{lcl} ud(x,\ell') & = & \{\ell \mid def(x,\ell) \wedge \exists \ell'' : (\ell,\ell'') \in flow(S_\star) \wedge clear(x,\ell'',\ell')\} \\ & \cup & \{? \mid clear(x, init(S_\star), \ell')\} \end{array}$$

where



- *def*(x,ℓ) means that the block ℓ assigns a value to x
- *clear*(x,ℓ,ℓ') means that none of the blocks on a path from ℓ to ℓ' contains an assignments to x but that the block ℓ' uses x (in a test or on the right hand side of an assignment)

ud chains - an alternative definition

$$\mathsf{UD} : \mathbf{Var}_\star \times \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Lab}_\star)$$

is defined by:

$$\mathsf{UD}(x,\ell) = \begin{cases} \{\ell' \mid (x,\ell') \in \mathsf{RD}_{entry}(\ell)\} & \text{if } x \in \mathit{gen}_{\mathsf{LV}}(B^\ell) \\ \emptyset & \text{otherwise} \end{cases}$$

One can show that:

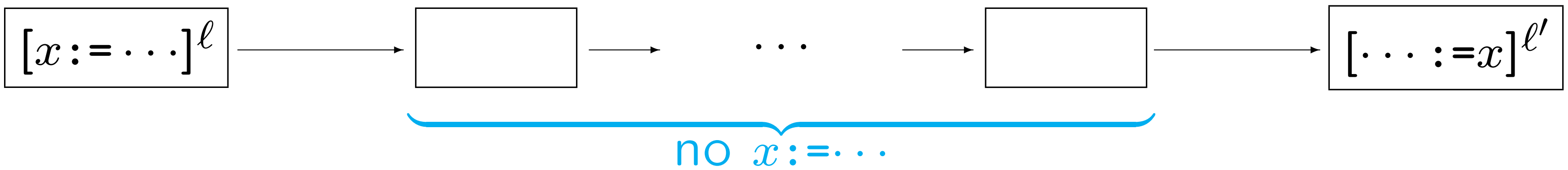
$$\mathit{ud}(x,\ell) = \mathsf{UD}(x,\ell)$$

du chains

$$du : \mathbf{Var}_\star \times \mathbf{Lab}_\star \to \mathcal{P}(\mathbf{Lab}_\star)$$

given by

$$du(x,\ell) = \begin{cases} \{\ell' \mid def(x,\ell) \wedge \exists \ell'' : (\ell,\ell'') \in flow(S_\star) \wedge clear(x,\ell'',\ell')\} \\ \qquad \text{if } \ell \neq \text{?} \\ \{\ell' \mid clear(x, init(S_\star), \ell')\} \\ \qquad \text{if } \ell = \text{?} \end{cases}$$



One can show that:

$$du(x,\ell) = \{\ell' \mid \ell \in ud(x,\ell')\}$$

Example:

$[\mathtt{x:=}0]^1; [\mathtt{x:=}3]^2; (\mathtt{if}\ [\mathtt{z=x}]^3\ \mathtt{then}\ [\mathtt{z:=}0]^4\ \mathtt{else}\ [\mathtt{z:=x}]^5); [\mathtt{y:=x}]^6; [\mathtt{x:=y+z}]^7$

$ud(x,\ell)$	x	y	z
1	$\emptyset$	$\emptyset$	$\emptyset$
2	$\emptyset$	$\emptyset$	$\emptyset$
3	$\{2\}$	$\emptyset$	$\{?\}$
4	$\emptyset$	$\emptyset$	$\emptyset$
5	$\{2\}$	$\emptyset$	$\emptyset$
6	$\{2\}$	$\emptyset$	$\emptyset$
7	$\emptyset$	$\{6\}$	$\{4,5\}$

$du(x,\ell)$	x	y	z
1	$\emptyset$	$\emptyset$	$\emptyset$
2	$\{3,5,6\}$	$\emptyset$	$\emptyset$
3	$\emptyset$	$\emptyset$	$\emptyset$
4	$\emptyset$	$\emptyset$	$\{7\}$
5	$\emptyset$	$\emptyset$	$\{7\}$
6	$\emptyset$	$\{7\}$	$\emptyset$
7	$\emptyset$	$\emptyset$	$\emptyset$
?	$\emptyset$	$\emptyset$	$\{3\}$

Theoretical Properties

- Structural Operational Semantics

- Correctness of Live Variables Analysis

The Semantics

A *state* is a mapping from variables to integers:

$$\sigma \in \mathbf{State} \;=\; \mathbf{Var} \to \mathbf{Z}$$

The semantics of arithmetic and boolean expressions

$$\mathcal{A} : \;\; \mathbf{AExp} \to (\mathbf{State} \to \mathbf{Z}) \quad \text{(no errors allowed)}$$

$$\mathcal{B} : \;\; \mathbf{BExp} \to (\mathbf{State} \to \mathbf{T}) \quad \text{(no errors allowed)}$$

The *transitions* of the semantics are of the form

$$\langle S, \sigma \rangle \to \sigma' \qquad \text{and} \qquad \langle S, \sigma \rangle \to \langle S', \sigma' \rangle$$

Transitions

$$\langle [x := a]^\ell, \sigma\rangle \to \sigma[x \mapsto \mathcal{A}[\![a]\!]\sigma]$$

$$\langle [\texttt{skip}]^\ell, \sigma\rangle \to \sigma$$

$$\frac{\langle S_1, \sigma\rangle \to \langle S_1', \sigma'\rangle}{\langle S_1; S_2, \sigma\rangle \to \langle S_1'; S_2, \sigma'\rangle}$$

$$\frac{\langle S_1, \sigma\rangle \to \sigma'}{\langle S_1; S_2, \sigma\rangle \to \langle S_2, \sigma'\rangle}$$

$$\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, \sigma\rangle \to \langle S_1, \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \mathit{true}$$

$$\langle \texttt{if } [b]^\ell \texttt{ then } S_1 \texttt{ else } S_2, \sigma\rangle \to \langle S_2, \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \mathit{false}$$

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, \sigma\rangle \to \langle (S; \texttt{while } [b]^\ell \texttt{ do } S), \sigma\rangle \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \mathit{true}$$

$$\langle \texttt{while } [b]^\ell \texttt{ do } S, \sigma\rangle \to \sigma \qquad \text{if } \mathcal{B}[\![b]\!]\sigma = \mathit{false}$$

Example:

$$\langle [\texttt{y:=x}]^1; [\texttt{z:=1}]^2; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{300}\rangle$$

$$\rightarrow \langle [\texttt{z:=1}]^2; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{330}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{331}\rangle$$

$$\rightarrow \langle [\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{331}\rangle$$

$$\rightarrow \langle [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{333}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{323}\rangle$$

$$\rightarrow \langle [\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{323}\rangle$$

$$\rightarrow \langle [\texttt{y:=y-1}]^5; \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{326}\rangle$$

$$\rightarrow \langle \texttt{while } [\texttt{y>1}]^3 \texttt{ do } ([\texttt{z:=z*y}]^4; [\texttt{y:=y-1}]^5); [\texttt{y:=0}]^6, \sigma_{316}\rangle$$

$$\rightarrow \langle [\texttt{y:=0}]^6, \sigma_{316}\rangle$$

$$\rightarrow \sigma_{306}$$

Equations and Constraints

Equation system $\mathsf{LV}^{=}(S_\star)$:

$$\mathsf{LV}_{exit}(\ell) = \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell',\ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) = (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}(B^\ell)) \cup \mathit{gen}_{\mathsf{LV}}(B^\ell)$$
where $B^\ell \in \mathit{blocks}(S_\star)$

Constraint system $\mathsf{LV}^{\subseteq}(S_\star)$:

$$\mathsf{LV}_{exit}(\ell) \supseteq \begin{cases} \emptyset & \text{if } \ell \in \mathit{final}(S_\star) \\ \bigcup\{\mathsf{LV}_{entry}(\ell') \mid (\ell',\ell) \in \mathit{flow}^R(S_\star)\} & \text{otherwise} \end{cases}$$

$$\mathsf{LV}_{entry}(\ell) \supseteq (\mathsf{LV}_{exit}(\ell) \backslash \mathit{kill}_{\mathsf{LV}}(B^\ell)) \cup \mathit{gen}_{\mathsf{LV}}(B^\ell)$$
where $B^\ell \in \mathit{blocks}(S_\star)$

Lemma

Each solution to the equation system $\mathsf{LV}^{=}(S_\star)$ is also a solution to the constraint system $\mathsf{LV}^{\subseteq}(S_\star)$.

Proof: Trivial.

Lemma

The least solution to the equation system $\mathsf{LV}^{=}(S_\star)$ is also the least solution to the constraint system $\mathsf{LV}^{\subseteq}(S_\star)$.

Proof: Use Tarski's Theorem.

Naive Proof: Proceed by contradiction. Suppose some LHS is strictly greater than the RHS. Replace the LHS by the RHS in the solution. Argue that you still have a solution. This establishes the desired contradiction.

Lemma

A solution *live* to the constraint system is preserved during computation

$$
\begin{array}{ccccccccc}
\langle S, \sigma_1 \rangle & \longrightarrow & \langle S', \sigma_1' \rangle & \longrightarrow & \cdots & \longrightarrow & \langle S'', \sigma_1'' \rangle & \longrightarrow & \sigma_1''' \\
\updownarrow \models \mathsf{LV}^{\subseteq} & & \updownarrow \models \mathsf{LV}^{\subseteq} & & & & \updownarrow \models \mathsf{LV}^{\subseteq} & & \\
\mathit{live} & & \mathit{live} & & \cdots & & \mathit{live} & &
\end{array}
$$

Proof: requires a lot of machinery — see the book.

Correctness Relation

$$\sigma_1 \sim_V \sigma_2$$

means that for all practical purposes the two states σ_1 and σ_2 are equal: only the values of the live variables of V matters and here the two states are equal.

Example:

Consider the statement $[\texttt{x:=y+z}]^\ell$

Let $V_1 = \{\texttt{y}, \texttt{z}\}$. Then $\sigma_1 \sim_{V_1} \sigma_2$ means $\sigma_1(\texttt{y}) = \sigma_2(\texttt{y}) \wedge \sigma_1(\texttt{z}) = \sigma_2(\texttt{z})$

Let $V_2 = \{\texttt{x}\}$. Then $\sigma_1 \sim_{V_2} \sigma_2$ means $\sigma_1(\texttt{x}) = \sigma_2(\texttt{x})$

Correctness Theorem

The relation "$\sim$" is *invariant* under computation: the live variables for the initial configuration remain live throughout the computation.

$$
\begin{array}{ccccccccc}
\langle S, \sigma_1 \rangle & \rightarrow & \langle S', \sigma_1' \rangle & \rightarrow & \cdots & \rightarrow & \langle S'', \sigma_1'' \rangle & \rightarrow & \sigma_1''' \\
\updownarrow \sim_V & & \updownarrow \sim_{V'} & & & & \updownarrow \sim_{V''} & & \updownarrow \sim_{V'''} \\
\langle S, \sigma_2 \rangle & \rightarrow & \langle S', \sigma_2' \rangle & \rightarrow & \cdots & \rightarrow & \langle S'', \sigma_2'' \rangle & \rightarrow & \sigma_2'''
\end{array}
$$

$V = \mathit{live}_{entry}(\mathit{init}(S))$

$V'' = \mathit{live}_{entry}(\mathit{init}(S''))$

$V' = \mathit{live}_{entry}(\mathit{init}(S'))$

$$
\begin{aligned}
V''' &= \mathit{live}_{exit}(\mathit{init}(S'')) \\
&= \mathit{live}_{exit}(\ell)
\end{aligned}
$$

for some $\ell \in \mathit{final}(S)$

Monotone Frameworks

- Monotone and Distributive Frameworks

- Instances of Frameworks

- Constant Propagation Analysis

The Overall Pattern

Each of the four classical analyses take the form

$$\begin{aligned} \mathit{Analysis}_\circ(\ell) &= \begin{cases} \iota & \text{if } \ell \in E \\ \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} & \text{otherwise} \end{cases} \\ \mathit{Analysis}_\bullet(\ell) &= f_\ell(\mathit{Analysis}_\circ(\ell)) \end{aligned}$$

where

- $\bigsqcup$ is $\bigcap$ or $\bigcup$ (and $\sqcup$ is $\cup$ or $\cap$),
- F is either $\mathit{flow}(S_\star)$ or $\mathit{flow}^R(S_\star)$,
- E is $\{\mathit{init}(S_\star)\}$ or $\mathit{final}(S_\star)$,
- ι specifies the initial or final analysis information, and
- f_ℓ is the transfer function associated with $B^\ell \in \mathit{blocks}(S_\star)$.

The Principle: forward versus backward

- The *forward analyses* have F to be *flow*$(S_\star)$ and then *Analysis*$_\circ$ concerns entry conditions and *Analysis*$_\bullet$ concerns exit conditions; the equation system presupposes that $S_\star$ has isolated entries.

- The *backward analyses* have F to be *flow*$^R(S_\star)$ and then *Analysis*$_\circ$ concerns exit conditions and *Analysis*$_\bullet$ concerns entry conditions; the equation system presupposes that $S_\star$ has isolated exits.

The Principle: union versus intersecton

- When $\bigsqcup$ is $\bigcap$ we require the greatest sets that solve the equations and we are able to detect properties satisfied by *all execution paths* reaching (or leaving) the entry (or exit) of a label; the analysis is called a must-analysis.

- When $\bigsqcup$ is $\bigcup$ we require the smallest sets that solve the equations and we are able to detect properties satisfied by *at least one execution path* to (or from) the entry (or exit) of a label; the analysis is called a may-analysis.

Property Spaces

The *property space*, L, is used to represent the data flow information, and the *combination operator*, $\bigsqcup$: $\mathcal{P}(L) \rightarrow L$, is used to combine information from different paths.

- L is a *complete lattice*, that is, a partially ordered set, $(L, \sqsubseteq)$, such that each subset, Y, has a least upper bound, $\bigsqcup Y$.

- L satisfies the *Ascending Chain Condition*; that is, each ascending chain eventually stabilises (meaning that if $(l_n)_n$ is such that $l_1 \sqsubseteq l_2 \sqsubseteq l_3 \sqsubseteq \cdots$,then there exists n such that $l_n = l_{n+1} = \cdots$).

Example: Reaching Definitions

- $L = \mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star)$ is partially ordered by subset inclusion so $\sqsubseteq$ is $\subseteq$
- the least upper bound operation $\bigsqcup$ is $\bigcup$ and the least element $\bot$ is $\emptyset$
- L satisfies the Ascending Chain Condition because $\mathbf{Var}_\star \times \mathbf{Lab}_\star$ is finite (unlike $\mathbf{Var} \times \mathbf{Lab}$)

Example: Available Expressions

- $L = \mathcal{P}(\mathbf{AExp}_\star)$ is partially ordered by superset inclusion so $\sqsubseteq$ is $\supseteq$

- the least upper bound operation $\bigsqcup$ is $\bigcap$ and the least element $\bot$ is $\mathbf{AExp}_\star$

- L satisfies the Ascending Chain Condition because $\mathbf{AExp}_\star$ is finite (unlike $\mathbf{AExp}$)

Transfer Functions

The set of transfer functions, $\mathcal{F}$, is a set of monotone functions over L, meaning that

$$l \sqsubseteq l' \text{ implies } f_\ell(l) \sqsubseteq f_\ell(l')$$

and furthermore they fulfil the following conditions:

- $\mathcal{F}$ contains *all* the transfer functions $f_\ell : L \to L$ in question (for $\ell \in \mathbf{Lab}_\star$)

- $\mathcal{F}$ contains the *identity function*

- $\mathcal{F}$ is *closed under composition* of functions

Frameworks

A *Monotone Framework* consists of:

- a complete lattice, L, that satisfies the Ascending Chain Condition; we write $\bigsqcup$ for the least upper bound operator
- a set $\mathcal{F}$ of monotone functions from L to L that contains the identity function and that is closed under function composition

A *Distributive Framework* is a Monotone Framework where additionally all functions f in $\mathcal{F}$ are required to be distributive:

$$f(l_1 \sqcup l_2) = f(l_1) \sqcup f(l_2)$$

Instances

An *instance* of a Framework consists of:

– the complete lattice, L, of the framework

– the space of functions, $\mathcal{F}$, of the framework

– a finite flow, F (typically *flow*$(S_\star)$ or *flow*$^R(S_\star)$)

– a finite set of *extremal labels*, E (typically $\{$*init*$(S_\star)\}$ or *final*$(S_\star)$)

– an *extremal value*, $\iota \in L$, for the extremal labels

– a mapping, $f_\cdot$, from the labels $\mathbf{Lab}_\star$ to transfer functions in $\mathcal{F}$

Equations of the Instance:

$$\mathit{Analysis}_\circ(\ell) = \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} \sqcup \iota_E^\ell$$
$$\text{where } \iota_E^\ell = \begin{cases} \iota & \text{if } \ell \in E \\ \bot & \text{if } \ell \notin E \end{cases}$$
$$\mathit{Analysis}_\bullet(\ell) = f_\ell(\mathit{Analysis}_\circ(\ell))$$

Constraints of the Instance:

$$\mathit{Analysis}_\circ(\ell) \sqsupseteq \bigsqcup\{\mathit{Analysis}_\bullet(\ell') \mid (\ell',\ell) \in F\} \sqcup \iota_E^\ell$$
$$\text{where } \iota_E^\ell = \begin{cases} \iota & \text{if } \ell \in E \\ \bot & \text{if } \ell \notin E \end{cases}$$
$$\mathit{Analysis}_\bullet(\ell) \sqsupseteq f_\ell(\mathit{Analysis}_\circ(\ell))$$

The Examples Revisited

	Available Expressions	Reaching Definitions	Very Busy Expressions	Live Variables
L	$\mathcal{P}(\mathbf{AExp}_\star)$	$\mathcal{P}(\mathbf{Var}_\star \times \mathbf{Lab}_\star)$	$\mathcal{P}(\mathbf{AExp}_\star)$	$\mathcal{P}(\mathbf{Var}_\star)$
$\sqsubseteq$	$\supseteq$	$\subseteq$	$\supseteq$	$\subseteq$
$\bigsqcup$	$\bigcap$	$\bigcup$	$\bigcap$	$\bigcup$
$\bot$	$\mathbf{AExp}_\star$	$\emptyset$	$\mathbf{AExp}_\star$	$\emptyset$
ι	$\emptyset$	$\{(x, ?) \mid x \in \mathit{FV}(S_\star)\}$	$\emptyset$	$\emptyset$
E	$\{\mathit{init}(S_\star)\}$	$\{\mathit{init}(S_\star)\}$	$\mathit{final}(S_\star)$	$\mathit{final}(S_\star)$
F	$\mathit{flow}(S_\star)$	$\mathit{flow}(S_\star)$	$\mathit{flow}^R(S_\star)$	$\mathit{flow}^R(S_\star)$
$\mathcal{F}$	$\{f : L \to L \mid \exists l_k, l_g : f(l) = (l \setminus l_k) \cup l_g\}$			
f_ℓ	$f_\ell(l) = (l \setminus \mathit{kill}(B^\ell)) \cup \mathit{gen}(B^\ell)$ where $B^\ell \in \mathit{blocks}(S_\star)$			

Bit Vector Frameworks

A *Bit Vector Framework* has

- $L = \mathcal{P}(D)$ for D finite
- $\mathcal{F} = \{f \mid \exists l_k, l_g : f(l) = (l \setminus l_k) \cup l_g\}$

Examples:

- Available Expressions
- Live Variables
- Reaching Definitions
- Very Busy Expressions

Lemma: Bit Vector Frameworks are always Distributive Frameworks

Proof

$$
\begin{array}{rcl}
f(l_1 \sqcup l_2) & = & \left\{ \begin{array}{l} f(l_1 \cup l_2) \\ f(l_1 \cap l_2) \end{array} \right. \\
& = & \left\{ \begin{array}{l} ((l_1 \setminus l_k) \cup (l_2 \setminus l_k)) \cup l_g \\ ((l_1 \setminus l_k) \cap (l_2 \setminus l_k)) \cup l_g \end{array} \right. \\
& = & \left\{ \begin{array}{l} f(l_1) \cup f(l_2) \\ f(l_1) \cap f(l_2) \end{array} \right.
\end{array}
\quad
\begin{array}{rl}
= & \left\{ \begin{array}{l} ((l_1 \cup l_2) \setminus l_k) \cup l_g \\ ((l_1 \cap l_2) \setminus l_k) \cup l_g \end{array} \right. \\
= & \left\{ \begin{array}{l} ((l_1 \setminus l_k) \cup l_g) \cup ((l_2 \setminus l_k) \cup l_g) \\ ((l_1 \setminus l_k) \cup l_g) \cap ((l_2 \setminus l_k) \cup l_g) \end{array} \right. \\
= & f(l_1) \sqcup f(l_2)
\end{array}
$$

- $id(l) = (l \setminus \emptyset) \cup \emptyset$
- $f_2(f_1(l)) = (((l \setminus l_k^1) \cup l_g^1) \setminus l_k^2) \cup l_g^2 = (l \setminus (l_k^1 \cup l_k^2)) \cup ((l_g^1 \setminus l_k^2) \cup l_g^2)$
- monotonicity follows from distributivity
- $\mathcal{P}(D)$ satisfies the Ascending Chain Condition because D is finite

The Constant Propagation Framework

An example of a Monotone Framework that is not a Distributive Framework

The aim of the *Constant Propagation Analysis* is to determine

> For each program point, whether or not a variable has a constant value whenever execution reaches that point.

Example:

$$[\mathtt{x}:=6]^1; [\mathtt{y}:=3]^2; \mathtt{while}\ [\mathtt{x} > \mathtt{y}]^3\ \mathtt{do}\ ([\mathtt{x}:=\mathtt{x}-1]^4; [\mathtt{z}:=\mathtt{y}*\mathtt{y}]^6)$$

The analysis enables a transformation into

$$[\mathtt{x}:=6]^1; [\mathtt{y}:=3]^2; \mathtt{while}\ [\mathtt{x} > 3]^3\ \mathtt{do}\ ([\mathtt{x}:=\mathtt{x}-1]^4; [\mathtt{z}:=9]^6)$$

Elements of L

$$\widehat{\mathbf{State}}_{\mathsf{CP}} = ((\mathbf{Var}_\star \to \mathbf{Z}^\top)_\bot, \sqsubseteq)$$

Idea:

- $\bot$ is the least element: no information is available
- $\widehat{\sigma} \in \mathbf{Var}_\star \to \mathbf{Z}^\top$ specifies for each variable whether it is constant:
  - $\widehat{\sigma}(x) \in \mathbf{Z}$: x is constant and the value is $\widehat{\sigma}(x)$
  - $\widehat{\sigma}(x) = \top$: x might not be constant

Partial Ordering on L

The partial ordering $\sqsubseteq$ on $(\mathbf{Var}_\star \to \mathbf{Z}^\top)_\bot$ is defined by

$$\forall \widehat{\sigma} \in (\mathbf{Var}_\star \to \mathbf{Z}^\top)_\bot : \quad \bot \sqsubseteq \widehat{\sigma}$$

$$\forall \widehat{\sigma}_1, \widehat{\sigma}_2 \in \mathbf{Var}_\star \to \mathbf{Z}^\top : \ \widehat{\sigma}_1 \sqsubseteq \widehat{\sigma}_2 \quad \underline{\text{iff}} \quad \forall x : \widehat{\sigma}_1(x) \sqsubseteq \widehat{\sigma}_2(x)$$

where $\mathbf{Z}^\top = \mathbf{Z} \cup \{\top\}$ is partially ordered as follows:

$$\forall z \in \mathbf{Z}^\top : z \sqsubseteq \top$$

$$\forall z_1, z_2 \in \mathbf{Z} : (z_1 \sqsubseteq z_2) \Leftrightarrow (z_1 = z_2)$$

Transfer Functions in $\mathcal{F}$

$$\mathcal{F}_{\mathsf{CP}} = \{f \mid f \text{ is a monotone function on } \widehat{\mathbf{State}}_{\mathsf{CP}}\}$$

Lemma

Constant Propagation as defined by $\widehat{\mathbf{State}}_{\mathsf{CP}}$ and $\mathcal{F}_{\mathsf{CP}}$ is a Monotone Framework

Instances

Constant Propagation is a forward analysis, so for the program $S_\star$:

- the flow, F, is *flow*$(S_\star)$,
- the extremal labels, E, is $\{$*init*$(S_\star)\}$,
- the extremal value, ι_{CP}, is $\lambda x.\top$, and
- the mapping, $f^{\mathsf{CP}}_{\cdot}$, of labels to transfer functions is as shown next

Constant Propagation Analysis

$$\mathcal{A}_{\mathsf{CP}} : \mathbf{AExp} \to (\widehat{\mathbf{State}}_{\mathsf{CP}} \to \mathbf{Z}^{\top}_{\bot})$$

$$\mathcal{A}_{\mathsf{CP}}[\![x]\!]\widehat{\sigma} = \begin{cases} \bot & \text{if } \widehat{\sigma} = \bot \\ \widehat{\sigma}(x) & \text{otherwise} \end{cases}$$

$$\mathcal{A}_{\mathsf{CP}}[\![n]\!]\widehat{\sigma} = \begin{cases} \bot & \text{if } \widehat{\sigma} = \bot \\ n & \text{otherwise} \end{cases}$$

$$\mathcal{A}_{\mathsf{CP}}[\![a_1\ op_a\ a_2]\!]\widehat{\sigma} = \mathcal{A}_{\mathsf{CP}}[\![a_1]\!]\widehat{\sigma}\ \widehat{\mathsf{op}}_a\ \mathcal{A}_{\mathsf{CP}}[\![a_2]\!]\widehat{\sigma}$$

transfer functions: f^{CP}_{ℓ}

$$[x := a]^{\ell} : \quad f^{\mathsf{CP}}_{\ell}(\widehat{\sigma}) = \begin{cases} \bot & \text{if } \widehat{\sigma} = \bot \\ \widehat{\sigma}[x \mapsto \mathcal{A}_{\mathsf{CP}}[\![a]\!]\widehat{\sigma}] & \text{otherwise} \end{cases}$$

$$[\mathtt{skip}]^{\ell} : \quad f^{\mathsf{CP}}_{\ell}(\widehat{\sigma}) = \widehat{\sigma}$$

$$[b]^{\ell} : \quad f^{\mathsf{CP}}_{\ell}(\widehat{\sigma}) = \widehat{\sigma}$$

Lemma

Constant Propagation is not a Distributive Framework

Proof

Consider the transfer function f_ℓ^{CP} for $[\mathtt{y:=x*x}]^\ell$

Let $\widehat{\sigma}_1$ and $\widehat{\sigma}_2$ be such that $\widehat{\sigma}_1(\mathtt{x}) = 1$ and $\widehat{\sigma}_2(\mathtt{x}) = -1$

Then $\widehat{\sigma}_1 \sqcup \widehat{\sigma}_2$ maps $\mathtt{x}$ to $\top$ — $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1 \sqcup \widehat{\sigma}_2)$ maps $\mathtt{y}$ to $\top$

Both $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1)$ and $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_2)$ map $\mathtt{y}$ to 1 — $f_\ell^{\mathsf{CP}}(\widehat{\sigma}_1) \sqcup f_\ell^{\mathsf{CP}}(\widehat{\sigma}_2)$ maps $\mathtt{y}$ to 1